SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

Friday 30th November 2012

Organised by the United Kingdom Mathematics Trust

The Senior Kangaroo paper allows students in the UK to test themselves on questions set for the best school-aged mathematicians from across Europe and beyond.

RULES AND GUIDELINES (to be read before starting):

1. Do not open the paper until the Invigilator tells you to do so.
2. Time allowed: **1 hour**.
3. The use of rough paper is allowed; **calculators** and measuring instruments are **forbidden**.
4. **Use B or HB pencil only** to complete your personal details and record your answers on the machine-readable Answer Sheet provided. **All answers are written using three digits, from 000 to 999.** For example, if you think the answer to a question is 42, write 042 at the top of the answer grid and then code your answer by putting solid black pencil lines through the 0, the 4 and the 2 beneath.

   Please note that the machine that reads your Answer Sheet will only see the solid black lines through the numbers beneath, not the written digits above. You must ensure that you code your answers or you will not receive any marks. There are further instructions and examples on the Answer Sheet.
5. The paper contains 20 questions. Five marks will be awarded for each correct answer. There is no penalty for giving an incorrect answer.
6. The questions on this paper challenge you **to think**, not to guess. Though you will not lose marks for getting answers wrong, you will undoubtedly get more marks, and more satisfaction, by doing a few questions carefully than by guessing lots of answers.

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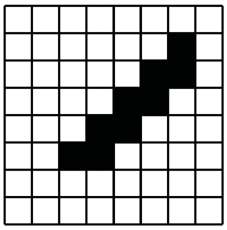
1. How many zeroes are there at the end of the number which is the product of the first 2012 prime numbers?

2. The size of the increase from each term to the next in the list a, $225\frac{1}{2}$, c, d, 284 is always the same. What is the value of a?

3. On the grid shown in the diagram, the shaded squares form a region, A.

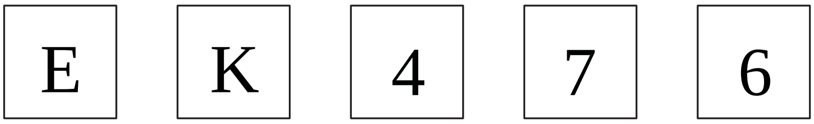


What is the maximum number of additional grid squares which can be shaded to form a region B such that B contains A and that the lengths of the perimeters of A and B are the same?

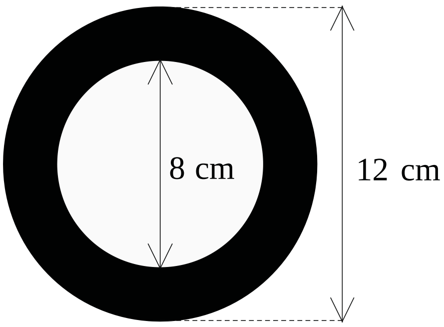
4. Five cards are laid on a table, as shown. Every card has a letter on one side and a number on the other side.

Peter says: “For every card on the table, if there is a vowel on one side of the card, then there is an even number on the other side.”

What is the smallest number of cards Sylvia must turn over in order to be certain that Peter is telling the truth?



5. Susan has two pendants made of the same material. They are equally thick and weigh the same. The first pendant is in the shape of an annulus created from two concentric circles, with diameters 8 cm and 12 cm, as shown. The shape of the second pendant is a disc. The diameter of the second pendant is written in the form $a\sqrt{b}$, where a is an integer and b is a prime integer.



What is the value of $a + b$?

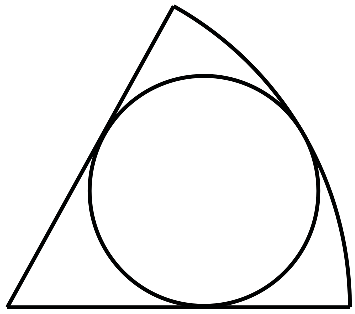
6. Given that $4^x = 9$ and $9^y = 256$, what is the value of xy?

7. When 1001 is divided by a single-digit number, the remainder is 5.
What is the remainder when 2012 is divided by the same single-digit number?

8. The three prime numbers a, b and c are such that $a > b > c$, $a + b + c = 52$ and $a - b - c = 22$.
What is the value of abc?

9. The diagram shows a circle touching a sector of another circle in three places. The ratio of the radius of the sector to the radius of the small circle is 3:1. The ratio of the area of the sector to the area of the small circle, written in its simplest form, is $p : q$.

What is the value of $p + q$?



10. Sixteen teams play in a volleyball league. Each team plays one game against every other team. For each game, the winning team is awarded 1 point, and the losing team 0 points. There are no draws. After all the games have been played and the teams have been ranked according to their total scores, the total scores form a sequence where the difference between consecutive terms is constant.

How many points did the team in first place receive?

11. Last year there were 30 more boys than girls in the school choir. This year the number of choir members has increased by 10%, the number of girls has increased by 20% and the number of boys by 5%.

How many members does the choir have this year?

12. The cells of a 4×4 grid are coloured black and white as shown in Figure 1. One move allows us to exchange the colourings of any two cells positioned in the same row or in the same column.

What is the minimum number of moves needed to obtain Figure 2?

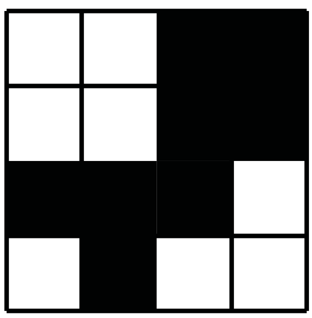


Figure 1

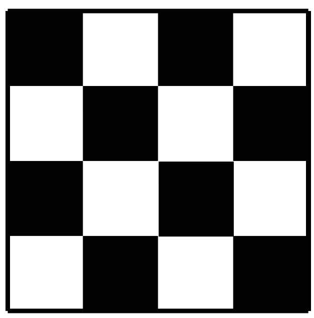
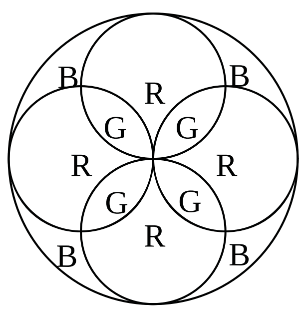


Figure 2

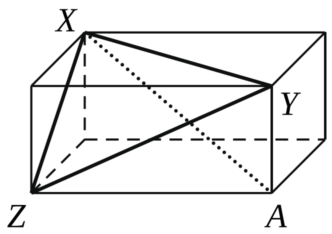
13. A circular stained-glass window is shown in the diagram. The four smaller circles are the same size and are positioned at equal intervals around the centre of the large circle. The letters R, G and B have been placed in regions of red, green and blue glass respectively. The total area of the green glass is 400.

What is the area of the blue glass?



14. The diagram shows a cuboid. In triangle XYZ, the lengths of XY, XZ and YZ are 9, 8 and $\sqrt{55}$ respectively.

What is the length of the diagonal XA shown?



15. The equation $x^2 - bx + 80 = 0$, where $b > 0$, has two integer-valued solutions. What is the sum of the possible values of b?

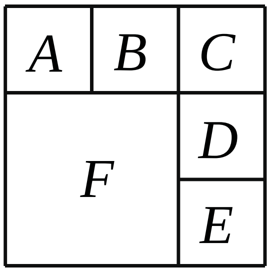
16. Given that $a + b = 5$ and $ab = 3$, what is the value of $a^4 + b^4$?

17. David removed one number from ten consecutive natural numbers. The sum of the remaining numbers was 2012.

Which number did he remove?

18. The diagram shows a square divided into six smaller squares labelled A, B, C, D, E and F. Two squares are considered to be adjacent if they have more than one point in common. The numbers 1, 2, 3, 4, 5 and 6 are to be placed in the smaller squares, one in each, so that no two adjacent squares contain numbers differing by 3.

How many different arrangements are possible?



19. A rectangle which has integer-length sides and area 36 is cut from a square with sides of length 20 so that one of the sides of the rectangle forms part of one of the sides of the square.

What is the largest possible perimeter of the remaining shape?

20. How many subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ exist in which the sum of the largest element and the smallest element is 11?

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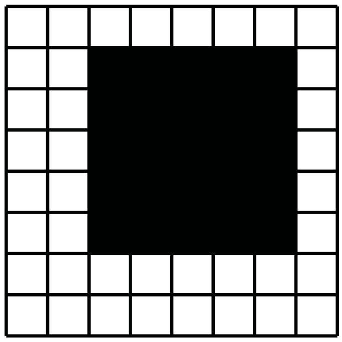
SOLUTIONS

2012 Senior Kangaroo Solutions

1. **1** In general, each zero at the end of an integer arises because, in the prime factorisation of the integer, there is a factor of 2 and a factor of 5 that can be paired to give a factor of 10. For example, 38 000 $= 2^4 \times 5^3 \times 19$ so 2 and 5 may be paired three times giving a factor of 1000. The product of the first 2012 prime numbers only contains a single factor of 2 and a single factor of 5 so there is only one zero at the end.

2. **206** Let the increase from one term to the next be i. From $225\frac{1}{2}$ to 284 the increase is $3i$ so $284 - 225\frac{1}{2} = 3i$. Therefore $3i = 58\frac{1}{2}$ and hence $i = 19\frac{1}{2}$. To find the value of a, we need to decrease $225\frac{1}{2}$ by i which gives $a = 206$.

3. **16** The diagram shows a region R, say, that certainly contains the region A, and has the same perimeter as A. We claim that R is the region with the largest possible area with these properties and so is B, the region required. To see this, observe that adding any number of extra grid squares to R will only increase the perimeter and so will not give a region of the type required. Therefore, the maximum number of additional grid squares that can be added is 16.



4. **2** For Sylvia to be certain that Peter is telling the truth, she must check that the card showing the letter E has an even number on the other side and that the card showing the number 7 does not have a vowel on the other side. The other three cards do not need to be checked since K is not a vowel and cards with even numbers on one side may have any letter on the other side.

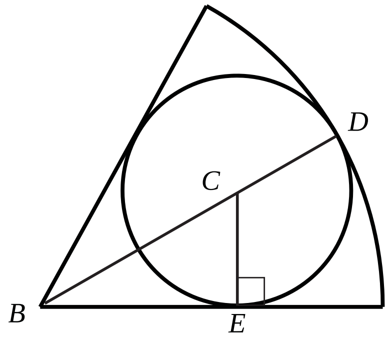
5. **9** For the weights of the two equally thick pendants to be the same, the area of the annulus must be equal to the area of the disc. The area of the annulus is $\pi \times 6^2 - \pi \times 4^2$ which is 20π. Since the area of a disc of radius r is πr^2, we have $\pi r^2 = 20\pi$ and so $r = \sqrt{20} = 2\sqrt{5}$. Thus the diameter of the second pendant is $2 \times 2\sqrt{5}$ which is $4\sqrt{5}$. So $a + b = 4 + 5 = 9$.

6. **4** We have $4^{xy} = \left(4^x\right)^y = 9^y = 256$. However $256 = 4^4$ so $xy = 4$.

7. **2** Let the single-digit number be k. Since the remainder is 5, k is larger than 5. Also k is a single-digit number so k is less than 10. Thus k is 6, 7, 8 or 9. However, the remainder when 1001 is divided by k is 5 so 996 is a multiple of k. Since 996 is not divisible by 7, 8 and 9 we conclude that $k = 6$. Finally, 2012 leaves a remainder of 2 when divided by 6.

8. **962** Adding the two given equations, we get $2a = 74$, which means $a = 37$. Substituting this value into the first equation we obtain $b + c = 15$. Therefore one of b and c is even and the other is odd. Since b and c are both prime, one of them is 2. This means the other is 13. Thus the product required is $2 \times 13 \times 37$, which is 962.

9. **5** Label the diagram as shown. Let the radius of the small circle be r, therefore $r = CD = CE$. We are told that the ratio of the radius of the sector to the radius of the small circle is 3 : 1 so $BD = 3 \times CD = 3r$. Therefore $BC = 2r$. Since triangle BCE is right-angled with two known sides, we have angle CBE is 30° and the area of the sector is a sixth of the area of a circle of radius $3r$. The ratio of the area of the sector to the area of the small circle is $\frac{1}{6}\pi(3r)^2 : \pi r^2$ which simplifies to $\frac{9}{6}\pi r^2 : \pi r^2$, this is, 3 : 2 in its simplest form. So $p + q = 5$.



10. **15** Each team played 15 matches so the maximum possible score for any one team is 15 points. Since the total score for a team must be an integer, the gap between consecutive team scores is also an integer. With 16 teams and no negative total scores, the gap is 1 point. This means the total scores are 15 points, 14 points, 13 points , …, 1 point, 0 points. Thus the team in first place scored 15 points, winning all their matches.

11. **99** Let the number of girls in the choir last year be x. This means there were $x + 30$ boys in the choir last year. This year there are 20% more girls, that is, $1.2x$ girls, and 5% more boys, that is, $1.05(x + 30)$ boys. The overall number in the choir this year is 10% more, that is, $1.1(x + x + 30)$. Putting these together, we get $1.1(2x + 30) = 1.2x + 1.05(x + 30)$. Multiplying out the brackets, we obtain $2.2x + 33 = 1.2x + 1.05x + 31.5$ and hence $15 = 0.05x$, so $x = 30$. The number in the choir this year is $2.2(2x + 30)$ which is therefore $2.2(2 \times 30 + 30)$. So there are 99 choir members this year.

12. **4** In the diagram, the squares labelled A, B, C, D, E and F all need to switch colour (from black to white or white to black, as appropriate). This could take as few as 3 exchanges, if the squares were distributed helpfully. Unfortunately, this is not the case. For example, each of A and F can only be paired with B. This means that it is impossible to use just three exchanges. So if we can perform the required switches in four exchanges, this must be the minimum number needed. This can be done in a number of ways, for example, exchange A and B, exchange C and D, exchange E and X, exchange X and F.

A			B
	C	D	
	E		X
			F

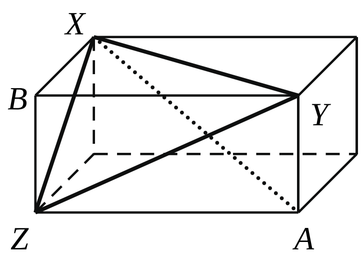
13. **400** Let the radius of each of the small circles be r. So the area of the whole window is $\pi(2r)^2$, which is $4\pi r^2$. Therefore $4\pi r^2 = 4(\text{R} + \text{G} + \text{B})$ and $\pi r^2 = \text{R} + \text{G} + \text{B}$. Now consider the area of one of the small circles. This is πr^2 but is also R + 2G. Equating these expressions for πr^2 gives R + G + B = R + 2G, which simplifies to B = G. Thus the area of the blue glass is equal to the area of the green glass.

Alternative: Note that the area of the large circle is equal to the area of four smaller circles since $\pi(2r)^2 = 4 \times \pi r^2$. But

the area of the window = the area of the 4 smaller circles − 4G + 4B.

Hence B = G.

14. **10** Let $BX = p$, $BY = q$ and $BZ = r$. Using Pythagoras' theorem in ΔBXY, ΔBXZ and ΔBYZ respectively, we obtain the equations: $p^2 + q^2 = 81$, $p^2 + r^2 = 64$ and $q^2 + r^2 = 55$. Adding all three equations, we get $2(p^2 + q^2 + r^2) = 81 + 64 + 55$ that is $p^2 + q^2 + r^2 = 100$. By applying Pythagoras' theorem to ΔBXZ and ΔAXZ, we find that the length of the diagonal XA is $\sqrt{p^2 + q^2 + r^2}$, which is $\sqrt{100} = 10$.



15. 186 The possible quadratic equations are $(x - 80)(x - 1) = 0$, $(x - 40)(x - 2) = 0$, $(x - 20)(x - 4) = 0$, $(x - 16)(x - 5) = 0$, $(x - 10)(x - 8) = 0$. These equations give the values of b as 81, 42, 24, 21 and 18 respectively. The sum of the possible values of b is 186.

16. 343 Expanding $(a + b)^4$ gives $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. We can rearrange this to give $a^4 + b^4 = (a + b)^4 - (4a^3b + 6a^2b^2 + 4ab^3)$ which can be written in the form $a^4 + b^4 = (a + b)^4 - (4ab(a^2 + b^2) + 6a^2b^2)$. Now using $a^2 + b^2 = (a + b)^2 - 2ab$ we can write $a^4 + b^4 = (a + b)^4 - (4ab((a + b)^2 - 2ab) + 6(ab)^2)$, that is $a^4 + b^4 = (a + b)^4 - 4ab(a + b)^2 + 2(ab)^2$. Substituting $a + b = 5$ and $ab = 3$, we obtain $a^4 + b^4 = 5^4 - (4 \times 3(5^2 - 2 \times 3) + 6 \times 3^2)$ which reduces to 343.

Note: For a quicker solution, observe that $a^2 + b^2 = (a + b)^2 - 2ab = 5^2 - 2 \times 3 = 19$ so that $a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2 = 19^2 - 2(ab)^2 = 361 - 18 = 343$.

17. 223 Let the smallest of the ten original numbers be x. The sum of all ten numbers is $x + (x + 1) + (x + 2) + \dots + (x + 9)$, which equals $10x + 45$.

Let the number removed be $x + a$ where $0 \leqslant a \leqslant 9$. Removing this number from the ten original numbers leaves 2012 so $10x + 45 - (x + a) = 2012$, that is, $9x = 1967 + a$. Dividing by 9, we get $x = 218\frac{5}{9} + \frac{a}{9}$. Since x is an integer, $a = 4$. Hence $x = 219$ and the number removed, $x + a$, is 223.

18. 96 The numbers 1, 2, 3, 4, 5 and 6 can be paired uniquely (1 with 4, 2 with 5, and 3 with 6) so that the difference between the numbers in each pair is 3. Any of the six numbers can be placed in position F but once that has been done, the partner of this number must be placed in position C as, otherwise, the square containing F would be adjacent to a square containing a number that differs by 3. Any of the remaining four numbers may be placed at position A, but then the number placed at B must be one of the two numbers from the pair that has so far been left unused. Finally, the two unused numbers may be positioned at D and E, in either order. Thus the number of possible arrangements is $6 \times 4 \times 2 \times 2$, which is 96.

19. 116 Let the lengths of the sides of the rectangle be a and b, where $a < b$. For the rectangle to be cut from the square, $b < 20$. But $ab = 36$ and a and b are integers, so the greatest possible value of b is 18. Note that cutting the rectangle so that it shares a corner with the original square and so two sides of the rectangle form part of two sides of the original square would still leave the remaining shape with a perimeter of 80. Cutting the rectangle from the square with one side of length a taken to be part of one of the sides of the square will give a perimeter of $80 + 2b$. Similarly, if the side of length b is taken to be part of one of the sides of the square, the perimeter of the new shape will be $80 + 2a$. For the largest perimeter, it is clearly better to do the former since a is less than b. Now $80 + 2b$ is largest when b is largest, so putting $b = 18$ we obtain the largest perimeter which is 116.

20. 341 If the smallest element was 1 then the largest element would be 10 (to give a sum of 11). Of the remaining possible elements, 2 could be included or not, 3 could be included or not and so on. This means that there are 2^8 possible subsets with smallest element 1 and largest element 10. Similarly there are 2^6 possible subsets with smallest element 2 and largest element 9, there are 2^4 possible subsets with smallest element 3 and largest element 8, there are 2^2 possible subsets with smallest element 4 and largest element 7, and there is one possible subset with smallest element 5 and largest element 6. In total this means there are $2^8 + 2^6 + 2^4 + 2^2 + 1 = 341$ possible subsets.